Coding and Counting
Arrangements
of Pseudolines

Seminar Geometry: Combinatorics and Algorithms
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point configuration

$(a, b) \in \mathbb{R}^2$
point configuration

\[ (a, b) \in \mathbb{R}^2 \]

... 1, 2, 3 \xrightarrow{12} 2, 1, 3 \xrightarrow{13} 2, 3, 1 \xrightarrow{23} 3, 2, 1 \ldots \] called *circular sequence*
Introduction / Duality

Point configuration

\[(a, b) \in \mathbb{R}^2\]

\[\rightarrow \text{ duality} \rightarrow\]

Line arrangement

\[\{(x, y) \in \mathbb{R}^2 \mid y = a \cdot x - b\}\]

...1, 2, 3 \xrightarrow{12} 2, 1, 3 \xrightarrow{13} 2, 3, 1 \xrightarrow{23} 3, 2, 1 \ldots \text{ called circular sequence}\]
point configuration 
\((a, b) \in \mathbb{R}^2\)  

\[ \text{duality} \]

line arrangement  
\[ \{ (x, y) \in \mathbb{R}^2 \mid y = a \cdot x - b \} \]

... 1, 2, 3 \[\rightarrow^{12}\] 2, 1, 3 \[\rightarrow^{13}\] 2, 3, 1 \[\rightarrow^{23}\] 3, 2, 1 ... called *circular sequence*
... 1, 2, 3 \xrightarrow{\text{swap}} \ldots \xrightarrow{\text{swap}} \ldots \xrightarrow{\text{swap}} 3, 2, 1 \ldots \quad \text{non-realizable called simple allowable sequence}
Fig. 1
An arrangement $A$ with a cell marked by a star and a wiring diagram of $A$. The local sequences of this arrangement are:

$\alpha_1 = 3, 5, 4, 6, 2,$
$\alpha_2 = 3, 4, 5, 6, 1,$
$\alpha_3 = 2, 1, 6, 5, 4,$
$\alpha_4 = 2, 5, 1, 6, 3,$
$\alpha_5 = 2, 4, 1, 6, 3,$
$\alpha_6 = 2, 1, 4, 5, 3.$

Fig. 2
Arrangement $A$ with its dual and the corresponding zonotopal tiling $Z$.

A simple zonotopal tiling $T$ is a tiling of a regular $2n$-gon with vertices $x_0, x_1, ..., x_{2n-1}$ in clockwise order starting with the highest vertex $x_0$. The tiles of $T$ are rhombi $R(i,j)$, 1 $\leq i < j \leq n$, such that $R(i,j)$ has one side which is a translated copy of the segment $[x_{i-1}, x_i]$ and one side which is a translated copy of the segment $[x_{j-1}, x_j]$. The tiles are not allowed to be rotated.

Simple zonotopal tilings can be viewed as normalized drawings of the duals of marked simple arrangements. Figure 2 shows an example. For additional information on zonotopal tilings and their relation to arrangements see [FV11] and [FV12].

Proofs of equivalence of the three representations are detailed in [FV11]. The basic tool for the proof of equivalence is to sweep a representation, resp. the arrangement, from left to right to transform one representation into another.

The Upper Bound
The upper bound for the number of simple Euclidean arrangements given in [FV11] was based on 'horizontal encodings' of arrangements. The first step was to replace the numbers in the local sequences $\alpha_i$ by single bits, a 1 for numbers $j$ with $j < i$ and a 0 for $j > i$.

The proof of Knuth [9] takes a 'vertical' approach. Let $A$ be an arrangement of $n+1$ pseudolines and consider pseudoline $n+1$ drawn into the wiring diagram of $A$.

\[\cdots 1, 2, 3 \xrightarrow{\text{swap}} \cdots \xrightarrow{\text{swap}} \cdots \xrightarrow{\text{swap}} 3, 2, 1 \cdots\]

called simple allowable sequence.

Examples of non-stretchable pseudoline arrangements

Non-pappus example (9 lines)

Pappus's hexagon theorem
Examples of non-stretchable pseudoline arrangements

The unrealizable pentagon (10 lines)
Pseudoline
\(x\)-monotone curve in the Euclidean plane

Arrangement of pseudolines
family of pseudolines such that every pair has a unique intersection

**Simple arrangement of pseudolines**
no three pseudolines have a common intersection point

**Marked** simple arrangement of pseudolines
designated unbounded cell, called the *north* cell
How many such arrangements are there?

When are two arrangements equal?

If we can map one to the other by homeomorphism of the plane

Wiring Diagram [G80]

canonical drawing

How many such arrangements are there?

When are two arrangements equal?

\[ B_n := \text{number of marked simple arrangements of } n \text{ pseudolines} \]

\[ B_1 = 1 \]

\[ B_2 = 1 \]

\[ B_3 = 2 \]

\[ B_4 = 8 \]

\[ B_5 = 62 \]

\[ B_6 = 908 \]

\[ \ldots \]

\[ B_{20} = ? \]

How many such arrangements are there?

When are two arrangements equal?

$B_n :=$ number of marked simple arrangements of $n$ pseudolines

$B_n = 2^{\Theta(n^2)}$

For comparison:

$L_n :=$ number of straight line arrangements

$L_n = 2^{O(n \log n)}$ [GP86]

How many such arrangements are there?

When are two arrangements equal?

\( B_n := \text{number of marked simple arrangements of } n \text{ pseudolines} \)

\[
B_n = 2^{\Theta(n^2)} = 2^{bn^2 + o(n^2)}
\]

- Conjecture: \( b = 1/2 \) [Knuth’92]
- \( b < 0.79, b > 1/6 \) [Knuth’92]
- \( b < \{0.72, 0.79, 0.69\} \) [Felsner’97]
- \( b < \{0.79, 0.66\}, b > 0.19 \) [F, Valtr’11]
- \( b > 1/9 \) [Matoušek’02]

How many such arrangements are there?

When are two arrangements equal?

$B_n := \text{number of marked simple arrangements of } n \text{ pseudolines}$

- Conjecture: $b = 1/2$ [Knuth’92]
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$B_n = 2^{\Theta(n^2)} = 2^{\Theta(n^2)} + o(n^2)$
$B_n := \text{number of marked simple arrangements of } n \text{ pseudolines}$

$\neq \text{counting allowable sequences}$

same pseudoline arrangement, different allowable sequence

Fig. 1


Simple zonotopal tilings can be viewed as normalized drawings of the duals of sequences of this arrangement are:

\[ A_n = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \]

Each \( \alpha_i \) is a vertex of a \( n \)-gon with one side which is a translated copy of \( \alpha_{i+1} \).

The upper bound for the number of simple Euclidean arrangements given in [FV11] was taken as support that the segment \( R(i,j) \) can be an arrangement of \( n \) pseudolines and consider pseudolines.

Proofs of equivalence of the three representations are detailed in [FV11]. The tiles are allowed to rotate.

The local sequences in clockwise order starting with the highest vertex \( \alpha_1 \) ≤ \( \alpha_2 \) ≤ \( \alpha_3 \) ≤ \( \alpha_4 \) ≤ \( \alpha_5 \) ≤ \( \alpha_6 \).

By single bits, a 1 for numbers less than or equal to 2, 3, 4, 5, 6.

\[ \text{Wiring Diagram} \]

\[ \text{Dual} \]

\[ \text{Zonotopal Tiling} \]

all figures: [FV11]
Wiring Diagram

Local Sequences
\[ \alpha_i = \text{order of the crossings with the lines } \{1, \ldots, n\} \setminus i \]
\[ \alpha_1 = 2, 3, 5, 4 \]
\[ \alpha_2 = 1, 5, 4, 3 \]
\[ \alpha_3 = 1, 5, 4, 2 \]
\[ \alpha_4 = 5, 1, 3, 2 \]
\[ \alpha_5 = 4, 1, 3, 2 \]

\[ B_n \leq ((n - 1)!)^n \]

Wiring Diagram

Local Sequences

\( \alpha_i = \text{order of the crossings with the lines } \{1, \ldots, n\}\setminus i \)

\( \alpha_1 = 2,3,5,4 \quad \tau_1 = 1,1,1,1 \)

\( \alpha_2 = 1,5,4,3 \quad \tau_2 = 0,1,1,1 \)

\( \alpha_3 = 1,5,4,2 \quad \tau_3 = 0,1,1,0 \)

\( \alpha_4 = 5,1,3,2 \quad \tau_4 = 1,0,0,0 \)

\( \alpha_5 = 4,1,3,2 \quad \tau_5 = 0,0,0,0 \)

\[ \downarrow \]

\( \tau_{i,j} = 1 \text{ iff } \alpha_{i,j} > i \)

\( B_n \leq ((n - 1)!)^n \quad B_n \leq 2^{n^2} \)

To do:
- Why are the $\alpha_i$ enough?
- Why are the $\tau_i$ enough?
- Count $\tau_i$ carefully

Local Sequences

$\alpha_i =$ order of the crossings with the lines \{1,\ldots,n\}\i

$\alpha_1 = 2,3,5,4$ $\tau_1 = 1,1,1,1$
$\alpha_2 = 1,5,4,3$ $\tau_2 = 0,1,1,1$
$\alpha_3 = 1,5,4,2$ $\tau_3 = 0,1,1,0$
$\alpha_4 = 5,1,3,2$ $\tau_4 = 1,0,0,0$
$\alpha_5 = 4,1,3,2$ $\tau_5 = 0,0,0,0$

$\tau_{i,j} = 1$ iff $\alpha_{i,j} > i$

$B_n \leq ((n - 1)!)^n$ $B_n \leq 2^{n^2}$
Wiring Diagram

**Algorithm:** search for the lowest neighboring pair and swap.

Local Sequences

<table>
<thead>
<tr>
<th>( \alpha_i )</th>
<th>( \tau_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 = 2,3,5,4 )</td>
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</tr>
<tr>
<td>( \alpha_2 = 1,5,4,3 )</td>
<td>( \tau_2 = 0,1,1,1 )</td>
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Algorithm: search for the lowest neighboring pair and swap.

\[ \alpha_1 = 2,3,5,4 \quad \tau_1 = 1,1,1,1 \]
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Wiring Diagram

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Algorithm: search for the lowest neighboring pair and swap.

Wiring Diagram

Local Sequences

\[
\begin{align*}
\alpha_1 &= 2,3,5,4 \\
\alpha_2 &= 1,5,4,3 \\
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\end{align*}
\]

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\tau_1 &= 1,1,1,1 \\
\tau_2 &= 0,1,1,1 \\
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Wiring Diagram

**Algorithm:** search for the lowest neighboring pair and swap.

Local Sequences

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Wiring Diagram

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Wiring Diagram

Algorithm: search for the lowest neighboring pair and swap.

Local Sequences

$\alpha_1 = 2,3,5,4$  \quad $\tau_1 = 1,1,1,1$

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$\alpha_5 = 4,1,3,2$  \quad $\tau_5 = 0,0,0,0$

Algorithm: search for the lowest neighboring pair and swap.

Local Sequences:

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\[\tau_1 = 1,1,1,1\]
\[\tau_2 = 0,1,1,1\]
\[\tau_3 = 0,1,1,0\]
\[\tau_4 = 1,0,0,0\]
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An arrangement partitions the plane into cells of dimensions 0, 1, or 2, the vertices, edges, and faces of the arrangement. The cells of an arrangement carry a natural lattice structure. Adding a 0 and a 1 element we obtain the face lattice of the arrangement. Two arrangements are considered to be isomorphic if their face lattices are isomorphic under the correspondence induced by some labeling.

Particularly nice pictures of arrangements of pseudolines are given by their wiring diagrams introduced in [5], see Fig. 1. Let $W$ be a wiring diagram of a simple arrangement of size $n$. For each abscissa $x$ where no crossing takes place the vertical order (upward) of the pseudolines at $x$ is a permutation $\pi_x$ of $\{1, 2, \ldots, n\}$. Assuming that no two crossings of $W$ have the same $x$ position we obtain $n^2$ different permutations. Denote by $\alpha$ the sequence of these permutations in left to right order. We note two properties of sequence $\alpha$:

1. The first element of $\alpha$ is the identity permutation $\alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_n$ and the last element of $\alpha$ is the reverse permutation $\alpha_n \rightarrow \cdots \rightarrow \alpha_2 \rightarrow \alpha_1$.

2. Two consecutive permutations in $\alpha$ differ by the reversal of an adjacent pair.

Following Goodman and Pollack [6], [7] we call a sequence $\alpha$ of $n^2$ permutations satisfying the above properties a simple allowable sequence. In general allowable sequences it is allowed for consecutive permutations to differ by the reversal of a larger substring. A simple allowable sequence is easily transformed into a wiring diagram and, hence, an arrangement of pseudolines. Note, however, that many allowable sequences may correspond to the same arrangement, see Fig. 2. Consecutive pairs of crossings that have no pseudoline in common can be interchanged without changing the arrangement.

Simple allowable sequences are basically the same as reflection networks, see [9]. Alternatively, they can also be seen as maximal chains in the weak Bruhat order of the symmetric group. In this last context their number $A_n$ has been determined by [F97].

Wiring Diagram

Initial Wiring Diagram

Wiring Diagram

Algorithm: search for the lowest ,,1,0“ pair and swap.

\[ \begin{align*}
\alpha_1 &= 2,3,5,4 & \tau_1 &= 1,1,1,1 \\
\alpha_2 &= 1,5,4,3 & \tau_2 &= 0,1,1,1 \\
\alpha_3 &= 1,5,4,2 & \tau_3 &= 0,1,1,0 \\
\alpha_4 &= 5,1,3,2 & \tau_4 &= 1,0,0,0 \\
\alpha_5 &= 4,1,3,2 & \tau_5 &= 0,0,0,0 
\end{align*} \]
**Wiring Diagram**

**Algorithm:** search for the lowest „1,0“ pair and swap.

\[
\begin{align*}
\alpha_1 &= 2,3,5,4 \\
\alpha_2 &= 1,5,4,3 \\
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\alpha_4 &= 5,1,3,2 \\
\alpha_5 &= 4,1,3,2
\end{align*}
\]

\[
\tau_1 = 1,1,1,1 \\
\tau_2 = 0,1,1,1 \\
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\tau_5 = 0,0,0,0
\]
Wiring Diagram

Algorithm: search for the lowest „1,0“ pair and swap.

Local Bit Sequences

\[ \alpha_1 = 2,3,5,4 \quad \tau_1 = 1,1,1,1 \]
\[ \alpha_2 = 1,5,4,3 \quad \tau_2 = 0,1,1,1 \]
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Wiring Diagram

Local Bit Sequences

Algorithm: search for the lowest ,,1,0“ pair and swap.

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\alpha_4 &= 5,1,3,2 & \tau_4 &= 1,0,0,0 \\
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\end{align*}
\]
Algorithm: search for the lowest "1,0" pair and swap.

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\end{align*} \]
Wiring Diagram

**Algorithm:** search for the lowest „1,0“ pair and swap.

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Wiring Diagram

**Algorithm:** search for the lowest „1,0“ pair and swap.

Local Bit Sequences

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Algorithm: search for the lowest „1,0“ pair and swap.

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Wiring Diagram

Algorithm: search for the lowest „1,0“ pair and swap.

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Wiring Diagram

Algorithm: search for the lowest „1,0“ pair and swap.

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Wiring Diagram

Algorithm: search for the lowest "1,0" pair and swap.

Local Bit Sequences

$\alpha_1 = 2,3,5,4$  \hspace{1cm} $\tau_1 = 1,1,1,1$

$\alpha_2 = 1,5,4,3$  \hspace{1cm} $\tau_2 = 0,1,1,1$

$\alpha_3 = 1,5,4,2$  \hspace{1cm} $\tau_3 = 0,1,1,0$

$\alpha_4 = 5,1,3,2$  \hspace{1cm} $\tau_4 = 1,0,0,0$

$\alpha_5 = 4,1,3,2$  \hspace{1cm} $\tau_5 = 0,0,0,0$

Algorithm: search for the lowest ,,1,0“ pair and swap.

\[
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\tau_2 &= 0,1,1,1 \\
\tau_3 &= 0,1,1,0 \\
\tau_4 &= 1,0,0,0 \\
\tau_5 &= 0,0,0,0
\end{align*}
\]
How many such local bit sequences $\tau_i$ can there be?

Observation: $\tau_i$ contains exactly $n - i$ ones.

$$|\mathcal{T}_n| = \binom{n-1}{0} \binom{n-1}{1} \binom{n-1}{2} \cdots \binom{n-1}{n-1}$$

$$|\mathcal{T}_n| = \frac{(n-1)^{n-1}}{(n-1)!} |\mathcal{T}_{n-1}| \leq \cdots \leq 2^{\left(\frac{1}{2} \log_2 e\right) \cdot n^2}$$

$$B_n \leq \left[\frac{1}{2} \log_2 e \right] = 0.7213$$
Local Bit Sequences

\( \tau_1 = 1,1,1,1 \)
\( \tau_2 = 0,1,1,1 \)
\( \tau_3 = 0,1,1,0 \)
\( \tau_4 = 1,0,0,0 \)
\( \tau_5 = 0,0,0,0 \)

Wiring Diagram

Log: 1,0,0,1,0

Wiring Diagram

Local Bit Sequences

\[
\begin{align*}
\tau_1 &= 1,1,1,1 \\
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\end{align*}
\]

Log: 1,0,0,1,0|1,1

Wiring Diagram

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Wiring Diagram

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Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0

Wiring Diagram

Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1

Local Bit Sequences

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Wiring Diagram

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Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0

Wiring Diagram

Local Bit Sequences

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\end{align*}
\]

Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0|0,0
Wiring Diagram

Local Bit Sequences

Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0|0,0|1

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Wiring Diagram

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Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0|0,0|1

Wiring Diagram

Local Bit Sequences

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Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0|0,0|1

We never log „,0,1“!
How many different logs are possible?

\[ |\mathcal{L}_n| \leq 2^n \cdot 3^{n(n-1)/2} = 2^{\log_2 3n^2 + O(n)} \]

\[ b \leq \frac{1}{2} \log_2 3 \approx 0.7924 \]

Log: 1,0,0,1,0|1,1|1,1|0,0|1,0|1,0|1|0|0,0|1

Log: 1,0,0,1,0| 2 | 2 | 0 | 1 | 1 | 1|0 | 0 | 1

\[ \tau_1 = 1,1,1,1X \]
\[ \tau_2 = 0,1,1,1X \]
\[ \tau_3 = 0,1,1,0X \]
\[ \tau_4 = 1,0,0,0X \]
\[ \tau_5 = 0,0,0,0X \]

Felsner’97

Read-Log of the $\tau_i$

Wiring Diagram

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Replace Matrix $A$

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<tr>
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<td>0</td>
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<td>0</td>
<td>0</td>
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</tbody>
</table>

Local Bit Sequences

$\tau_1 = 1,1,1,1,1$
$\tau_2 = 0,1,1,1,1$
$\tau_3 = 0,1,1,0,1$
$\tau_4 = 1,0,0,0,1$
$\tau_5 = 0,0,0,0,0$

If we read from $\tau_i$ and $\tau_j$, we write to $a_{i,j}$ and $a_{j,i} \sim a_{i,j} \geq a_{j,i}$

Replace Matrix satisfies both conditions $\sim b < 0.6974$

Different approach: insert a new pseudoline as a vertical cutpath

\[ \gamma_n: \text{maximal number of cutpaths for } n \text{ pseudolines} \]

\[ B_{n+1} \leq \gamma_n \cdot B_n \implies \text{goal: bound } \gamma_n \]

Counting cutpaths

Naive: $n^n$  Clever: $3^n$

Lemma (Knuth):
Any cutpath $p$ sees any pseudoline $j$ at most once as a middle.

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Any cutpath $p$ sees any pseudoline $j$ at most once as a middle.

$p$: exactly 3 wire changes
$j$: at least 5 wire changes

crossing $j \Rightarrow$ crossing $p$

a contradiction.
Encoding the cutpath

- $M \in \{0,1\}^n$, subset of middles used
- $\beta \in \{0,1\}^n$, sequence of left turns used

Logging the steps

- cross 2 as a middle
- right turn
- left turn
- left turn
- unique successor

Encoding the log

- $M \in \{0,1\}^n$, subset of middles used
  $M = [0,1,0,0,0]$  
- $\beta \in \{0,1\}^n$, sequence of left turns used
  $\beta = [0,0,1,1,0]$

Encoding the cutpath:

- $M \in \{0,1\}^n$, subset of middles used
- $\beta \in \{0,1\}^n$, sequence of left turns used

$B_{n+1} \leq \gamma_n \cdot B_n \rightarrow \gamma_n \leq 2^n \cdot 2^n = 4^n \sim B_n \leq 2^{2\sum_{i=1}^{n} i} \rightarrow b \leq 1$

Observations:

- Let $k = \#middles$ used
- For every 1-bit in $M$, some bit in $\beta$ is irrelevant
- Lookups in $\beta$ are done in increasing order

$\rightarrow \gamma_n \leq \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = 2^n \left(1 + \frac{1}{2}\right)^n = 3^n \sim b \leq \frac{1}{2} \log_2 3 = 0.7924$

$M = [0,1,0,0,0]$

$\beta = [\times,0,1,1,0]$
Encoding the cutpath:

- \( M \in \{0,1\}^n \), subset of middles used
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\[
B_{n+1} \leq \gamma_n \cdot B_n \Rightarrow \gamma_n \leq 2^n \cdot 2^n = 4^n \Rightarrow B_n \leq 2^\sum_{i=1}^{n} i \Rightarrow b \leq 1
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Observations:

- Let \( k = \#\) middles used
- For every 1-bit in \( M \), some bit in \( \beta \) is irrelevant
- Lookups in \( \beta \) are done in increasing order

\[
\gamma_n \leq \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} 2^{n-k} = 2^n \left( 1 + \frac{1}{2} \right)^n = 3^n \Rightarrow b \leq \frac{1}{2} \log_2 2.5 = 0.6609
\]

\[M = [0,1,0,0,0]\]
\[\beta = [\text{x},0,1,1,0]\]
Allowing many pseudoline arrangements

\[ \frac{n}{3} \]

Allowing many pseudoline arrangements

\[ \frac{n}{3} \] lines

number of choices between \( \frac{n}{6} \) and \( \frac{n}{3} \)

\[ B_n \geq 2^{\frac{n}{3} \cdot \frac{n}{4.5}} \]

Allowing many pseudoline arrangements

\[ B_n \geq 2^{\frac{n}{3}} \cdot \frac{n}{4.5} \]

Recurring:

\[ B_n \geq 2^{\frac{n}{3}} \cdot \frac{n}{4.5} \cdot \left( 2^{\frac{n}{9}} \cdot \frac{n}{4.5 \cdot 3} \right)^3 \ldots \]

\[ B_n \geq 2^{\frac{n^2}{4.5}} \cdot \sum_{i=1}^{\infty} \left( \frac{1}{3} \right)^i = 2^{\frac{n^2}{4.5}} \cdot \frac{1}{2} = 2^{\frac{n^2}{9}} \]

Knuth’s conjecture still stands: $b = 0.5$?

Current state: $0.1887 < b < 0.657$

Conclusion

Summary

Knuth’s conjecture still stands: $b = 0.5$?

Current state: $0.1887 < b < 0.657$